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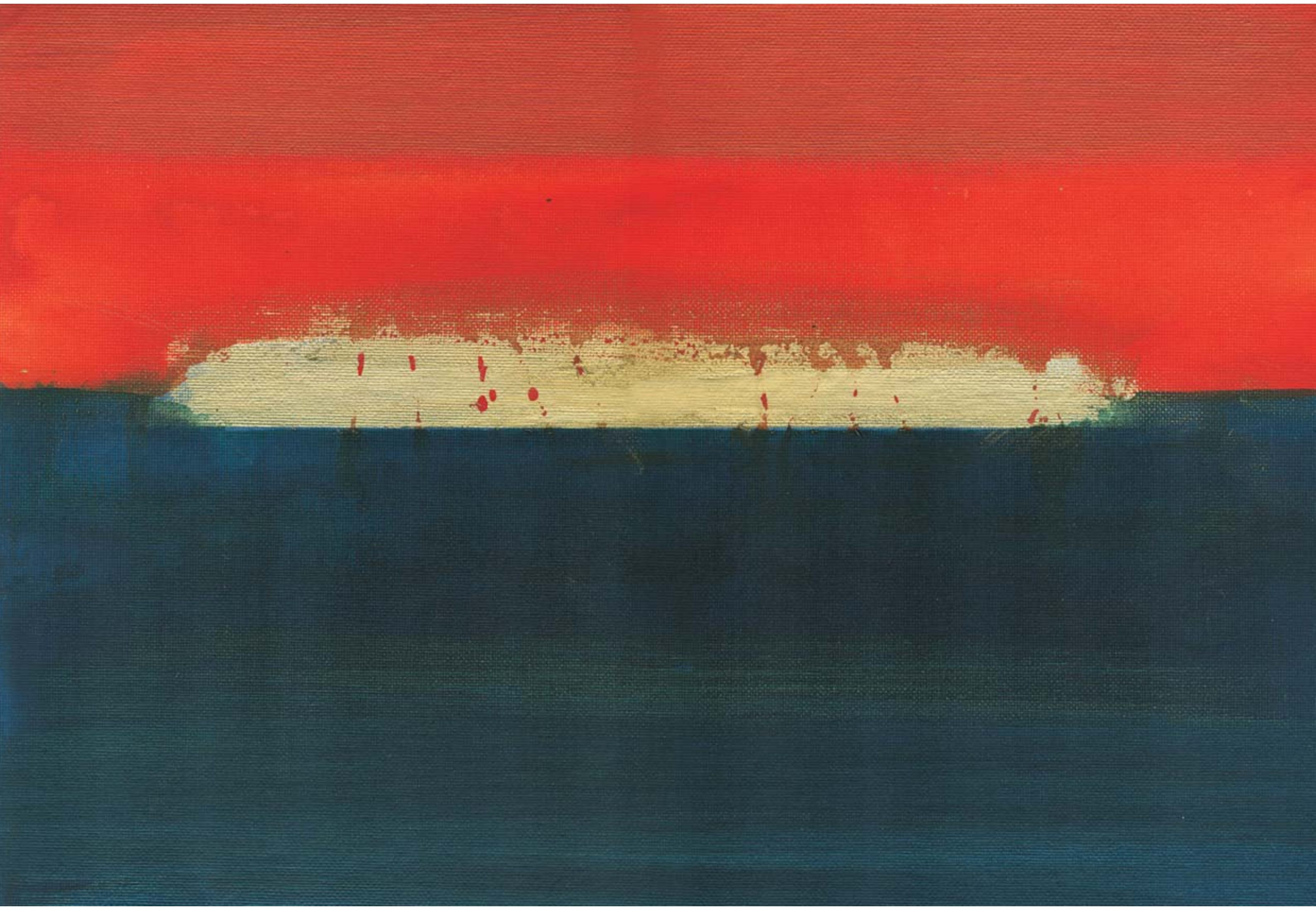
A TERM STRUCTURE OF INTEREST RATES MODEL WITH ZERO LOWER BOUND AND THE EUROPEAN CENTRAL BANK'S NON-STANDARD MONETARY POLICY MEASURES



CONTENTS

ABBREVIATIONS

ABSPP – asset-backed securities purchase programme
APP – expanded asset purchase programme
CBPP – covered bond purchase programme
CBPP2 – second covered bond purchase programme
CBPP3 – third covered bond purchase programme
DSGE – dynamic stochastic general equilibrium model
ECB – European Central Bank
GDP – gross domestic product
IMF – International Monetary Fund
LTRO – longer-term refinancing operation
MBS – mortgage-backed security
OIS – overnight indexed swap
PSPP – public sector purchase programme
SMP – Securities Markets Programme
TLTRO – targeted longer-term refinancing operation
WEO – World Economic Outlook
ZLB – zero lower bound

ABSTRACT

This paper proposes a ZLB/shadow rate term structure of interest rates model with both unobservable factors and those of non-standard monetary policy measures. The non-standard factors include the ECB's holdings of APP and LTROs as well as their weighted average maturities. The model is approximated by the Taylor series expansion and estimated by the extended Kalman filter, using the sample from July 2009 to September 2015. The results show that the 5-year OIS rate at the end of September 2015 was about 60 basis points lower than it would have been in the case of the absence of the non-standard monetary policy measures.

Keywords: term structure of interest rates, lower bound, non-linear Kalman filter, non-standard monetary policy measures.

JEL codes: C24, C32, E43, E58, G12

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The views expressed in this paper are those of the author, who is an employee of the Monetary Policy Department of Latvijas Banka, and do not necessarily reflect the stance of Latvijas Banka. The author assumes sole responsibility for any errors and omissions. E-mail address: Viktors.Ajevskis@bank.lv.

1. INTRODUCTION

In response to the crisis, many central banks have reduced their policy interest rates to their effective lower bound. Under these circumstances, central banks have no room for further monetary easing in the traditional way, i.e. by lowering their policy rate. For this reason, central banks introduce a set of non-standard monetary policy measures, which typically include LTROs, the APP and forward guidance.

There are several approaches to estimate the extent to which these programmes affect longer-term interest rates. The first strand of the literature relies on the event study methodology to determine the effect of the APP on interest rates at the onset of the programmes (Gagnon et al. (2011), Krishnamurthy and Vissing-Jørgensen (2011) and Swanson (2011)). The second approach employs a reduced-form regression analysis to estimate the effect of the APP on interest rates over time (D'Amico and King (2013), D'Amico et al. (2012) and Meaning and Zhu (2011)). The third strand incorporates changes in the net private supply of different assets into a structural model of the yield curve (Li and Wei (2012) and Ihrig et al. (2012)).

Li and Wei (2012) extend the standard Gaussian affine no-arbitrage term structure model to allow the Treasury and MBS to supply variables to affect term premiums. They estimate the model, using the data from March 1994 to July 2007 on Treasury yields as well as supply and maturity characteristics of private Treasury and MBS holdings, then they apply estimated coefficients to assess the term premium effect of the APP. However, their approach could be considered as not fully consistent because they use an affine type model that may be incorrect at the ZLB as shown in Krippner (2013) and Christensen and Rudebusch (2013). Moreover, the use of data referring to a period prior to 2008 for estimation might be inappropriate as the sensitivity coefficients of asset supply in equations, which determine yields, can be different before and after the crisis. Furthermore, asset holdings by the Federal Reserve System of the United States were much smaller before the crisis than after it. Thus, this approach may be vulnerable to the Lucas critique since non-stantard policy measures can be treated as structural monetary policy changes.

Altavilla et al. (2015) evaluate the effects of the ECB's APP on asset prices. They use the event study methodology and find that the APP has significantly lowered yields for a broad set of market segments, e.g. yields on long-term sovereign bonds decline by approximately 30–50 basis points at the 10-year maturity for the implied euro area term structure. They find that the effects of the APP generally rise with maturity and riskiness of assets. Altavilla et al. (2015) assess the APP main transmission channels and give the explanation of the obtained results by developing a term structure model with bond supply effects accounting for assets with different types of risks.[1] However, the explanations are only qualitative in character as they do not estimate their model. Moreover, the model does not take into account the interest lower bound.

To deal with the ZLB in constructing a term structure of interest rates models, Black (1995) proposed a concept of the shadow rate, i.e. a rate that equals the short interest rate when it is positive and may be negative when the short rate equals zero. In recent years, this approach has been renewed by Krippner (2013) and employed by various authors. Most of them use a specification with two or three factors and

[1] This approach is based on the model developed by Vayanos and Vila (2009).

approximations based on either option pricing as in Krippner (2013), Bauer and Rudebusch (2015), Christensen and Rudebusch (2013) or the Taylor series expansion as in Priebsch (2013), Wu and Xia (2014), Lemke and Vladu (2014)).

The contribution of the paper is as follows. First, we develop a term structure model that includes the size and duration of the ECB's balance sheet as deterministic observable factors. Second, the model takes into account the lower bounds of interest rates and is approximated by the Taylor series expansion of order two. Third, we estimate the model by using the extended Kalman filter and the OIS data from July 2009 to September 2015. Fourth, based on the estimated model we carry out a counterfactual simulation to evaluate the effect of the ECB's non-stantard monetary policy measures on the long-term OIS yields. Our estimations show that the non-standard monetary measures had reduced the 5-year OIS rate about 60 basis points by September 2015. We estimate the shadow rate as a by-product.

2. THE MODEL

2.1 The model setup

We assume that yields are driven by two unobservable factors $X_t^1 = [X_{1,t}^1; X_{2,t}^1]$ that follow a first-order vector autoregressive process:

$$X_{t+1}^1 = \left(\mathrm{I} - \Phi^{\mathbb{P}}\right)\mu^{\mathbb{P}} + \Phi^{\mathbb{P}} X_t^1 + \Sigma\varepsilon_{t+1}, \varepsilon_{t+1} \sim N(0, I_N) \quad (1)$$

and by two factors related to the ECB's balance sheet, namely, the euro amount of programmes and operations for the non-standard monetary policy measures X_t^a and their average duration X_t^d. We denote the vector of this factor $X_t^2 = [X_t^a, X_t^d]$ and define X_t^a as a deterministic process that at each time t is equaled to the ratio of the sum of the ECB's APP holdings and LTROs to the euro area nominal GDP. The future values of this process at the times $t + u, u = 1,2, \dots$ are assumed to equal our projection for these holdings. When a new programme starts, its overall path is known and the expected path of X_{t+u}^2 changes. Regarding this path, investors expect the ECB to start selling securities after the end of a programme. X_t^d is a corresponding weighted duration of ECB asset holdings and LTROs. The construction of the time series for these processes is described in Subsection 3.1.

The shadow rate s_t is defined as

$$s_t = \delta_0 + \delta_1 X_t^1 + \delta_2 X_t^2 \quad (2)$$

and the observed short interest rate as the censored shadow rate

$$r_t = max(s_t, r^{LB}) \quad (3),$$

where r^{LB} is a lower bound, which is not necessarily equaled to zero. The market prices of risk Λ_t are given as linear functions of the unobservable factors

$$\Lambda_t = \lambda_0 + \lambda_1 X_t^1 \quad (4).$$

The observable factors X_t^2 are not included in the market price of risk because of the assumption of their deterministic nature. It is a challenging task to put stochastics into the observable factors or, in other words, to choose a data generation process for X_t^2, taking into account the historical and future projection paths of the size of the ECB's balance sheet and its average duration (see Figures 1–3). For the purpose of simplicity, we make a perfect foresight assumption for these processes.

Figure 1

Time series of the APP and longer-term liquidity operations from July 2009 to September 2015 and their projections after September 2015

(in billions of euro)

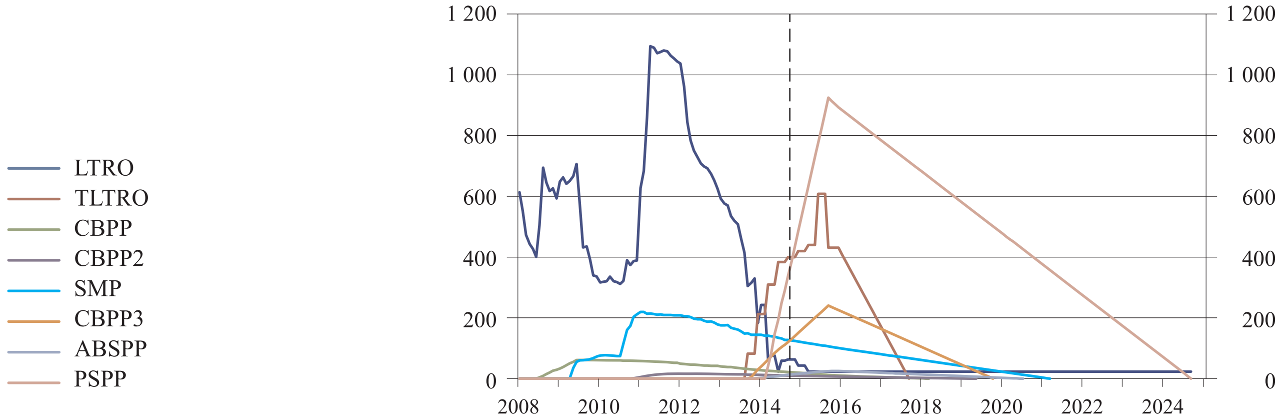


Figure 2

Actual aggregate ECB's balance sheet (from July 2009 to September 2015) and its projection (after September 2015) relative to the euro area nominal GDP

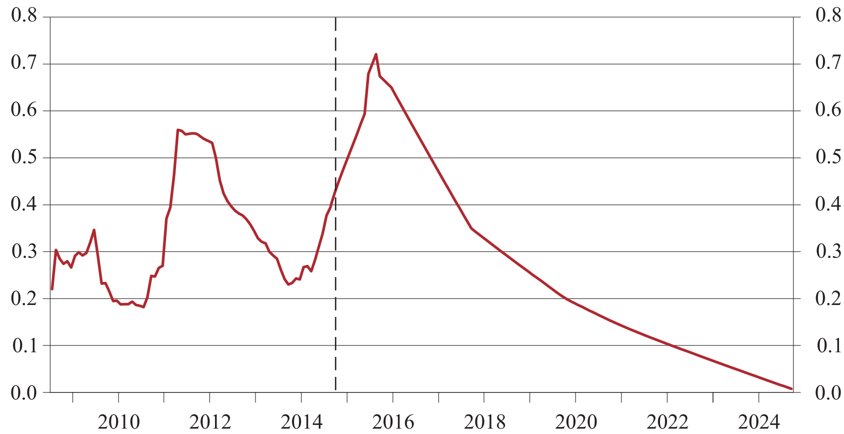
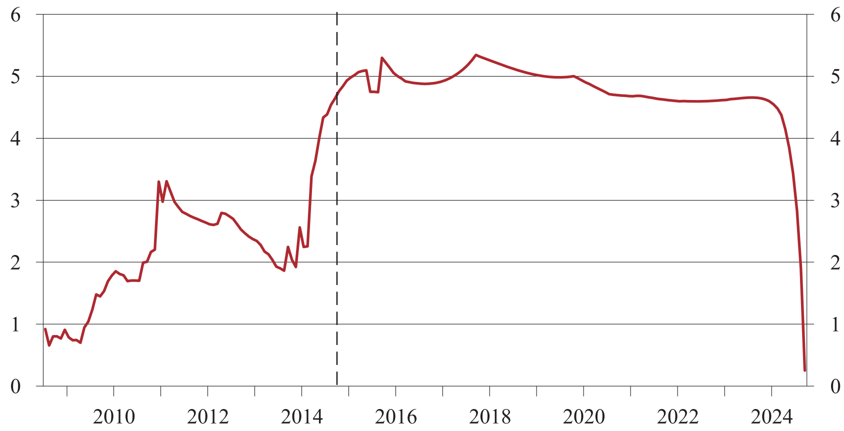


Figure 3

Actual average duration of the ECB's balance sheet (from July 2009 to September 2015) and its projection (after September 2015) in relation to the non-standard monetary measures

(years)



The price of an n-period zero-coupon bond P_t^n is given by

$$P_t^n = E_t^{\mathbb{Q}} \exp\left(-\sum_{i=0}^{n-1} r_{t+i}\right) \qquad (5),$$

where E_t^Q is expectations under the risk-neutral measure. The dynamics of the factors under this measure is:

$$X_{t+1}^1 = (1-\Phi^{\mathbb{Q}})\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t^1 + \Sigma\varepsilon_{t+1}, \varepsilon_{t+1} \sim N(0, I_N) \qquad (6),$$

where $\mu^{\mathbb{Q}}$ and $\Phi^{\mathbb{Q}}$ satisfy

$$\mu^{\mathbb{Q}} = \mu^{\mathbb{P}} - \Sigma\lambda_0 \qquad (7)$$

$$\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma\lambda_1 \qquad (8).$$

The bond price formula (5) has the form:

$$P_t^n = E_t^{\mathbb{Q}} \exp\left(\sum_{i=0}^{n-1} \max(s_{t+i}, r^{LB})\right) \qquad (9).$$

The lower bound implies that yields lose the affine structure and are represented by non-linear mapping

$$y_t^n = g_n\left(X_t, \theta, r^{LB}\right) = -\frac{1}{n} \ln\left\{E_t^{\mathbb{Q}} \exp\left(\sum_{i=0}^{n-1} \max(s_{t+i}, r^{LB})\right)\right\} \qquad (10),$$

where $X_t = [X_t^1, X_t^2]'$ and θ is a vector of parameters. The mapping (10) does not have an analytical form if the dimension of state variables is more than one. For this reason, some approximation of g_n is necessary to deal with the problem of computing the yields. We approximate g_n by expanding bond prices in formula (9) in a Taylor's series up to order two (for details see Subsections A1–A3 of the Appendix). It follows from (4) and (10) that the bond prices depend not only on the current values of the ECB's balance sheet factor but also on their expected future path.

2.2 Transmission channels

Large-scale asset purchases by central banks are supposed to affect financial market prices via two main channels. First, through the portfolio balance channel by which investors relocate their portfolios owing to liquidity generated by asset purchases, thus affecting prices in various market segments. Second, in relation to the signalling channel, the expansion in the size of the balance sheet lowers expectations of the future policy rate. This results in expectations of a looser monetary policy stance in the future and, in turn, eases the current monetary policy stance.

The portfolio balance channel consists of two subchannels that reduce the term premium. The first one is the duration channel that lowers the duration risk remaining in the private sector. If a central bank purchases long-term bonds, the maturity of the remaining bond portfolio decreases, thus reducing its risk. This may engage the banking sector to take more risk by giving loans, which are riskier assets, to firms and households. Through the scarcity channel, the central bank creates a scarcity in the assets it purchases. If investors have preferred habitats for purchased bond maturities, the bond prices increase and, therefore, their yields drop.

The proposed modelling framework primarily captures the signal channel, i.e. non-standard policy measures affect long-term yields through expectations of the short-term interest rate. The duration and scarcity channels work through short interest rate expectations that depend on the ECB's balance sheet and its duration in a quite non-linear way owing to the lower bound. Note also that in the model, the LTROs have the same effect on the short-term interest rate as asset purchases.

3. DATA AND PROJECTIONS OF PORTFOLIOS

3.1 Data

The interest rate data are the end-of-the-month yields on the OIS for maturities of 3 and 6 months and 1, 2, 3, 4, 5 and 10 years. The sample spans from July 2009 (the launch of the CBPP) to September 2015. The choice of such a short sample period has an advantage of less vulnerability to the Lucas critique. Indeed, the non-standard policy measures were never implemented by the ECB before the crisis. The use of reduced-form estimations based on the data prior to the introduction of the unconventional monetary policy for evaluating the effects of this policy (as done in Li and Wei (2012) and Hamilton and Wu (2012)) could be inappropriate. For instance, let us consider the construction of the term structure of interest rates in the DSGE framework (see, e.g. Hördahl et al. (2006)). In this case, the loadings for yields are functions of deep parameters and the Taylor rule coefficients.[2] When the short-term interest rate reaches the ZLB, the Taylor rule switches off, and thus its coefficients dissapear in the yield loadings. Because of this, the loadings for yields must be different for the normal times and for the times of the ZLB.

In contrast to Li and Wei (2012), we use data on the ECB's holdings but not on private asset holdings. As the relation between the shadow rate and observable factors are linear, these two approaches are equivalent. The difference is only in the signs of the respective coefficients, i.e. the coefficients of the ECB's asset holdings and their average maturity should have negative signs in equation (2).

At each time t we define X_t^a as a deterministic process that is equaled to the ratio of the sum of the security holdings by the ECB's APP programmes and longer-term liquidity operations to the euro area nominal GDP. This sum consists of the following items of the ECB's asset side of the balance sheet: LTROs, TLTROs, the CBPP, SMP, CBPP2, CBPP3, ABSPP and PSPP (for more details see the Appendix). At each time t the future values of each of these variables equal the actual holdings before September 2015 and our projection, described below, after September 2015. The same is valid for GDP, for which the 5-year projection beyond April 2015 is provided from the IMF WEO, April 2015. We then assume that GDP growth from January 2020 to September 2025 is 1.5%. To obtain monthly data for GDP, we use linear interpolation of quarterly data. The process X_t^a is also a deterministic one that is defined as the respective weighted maturity of X_t^a from July 2009 to September 2015 and its projection after September 2015.

3.2 Projections of the ECB's balance sheet items

Up to September 2015, we use the actual end-of-month ECB balance sheet data for all portfolios and tender operations. As of August 2015, we project the ECB's balance sheet items based on the following assumptions:

[2] The persistence coefficients and standard deviations of exogenous shocks as well.

1. For the portfolios of the terminated programmes (the CBPP, CBPP2 and SMP), we assume that the balances will continue to decline in a linear fashion. We derive the rate of maturing securities from the historical data and assume that the same rate will persist.

2. For LTROs, we take into consideration the substitution effect of the TLTRO programme and reduce the future uptake by the amount of the TLTRO programme uptake (the total balance of LTROs and TLTROs remains unchanged until the penultimate TLTRO in March 2016).

3. For TLTROs, we assume that the take-up in the following three operations will decrease from the previous ones as the majority of the banks that wanted to take advantage of the cheap long-term financing offered by the ECB have already seized it. We employ an expert judgment which, according to a conservative scenario, foresees the uptake of 20 billion euro in each of the remaining three out of four TLTROs through March 2016. We expect a higher take-up in the last TLTRO in June 2016 as banks will try to prefinance the higher costing loans from the first two TLTROs which will be subject to repayment in September 2016 when we expect banks to repay about 30% of the TLTRO loans.

4. For the purchases under the APP, we assume that purchases totalling 60 billion euro per month will be made through September 2016 in line with a communication by the Governing Council of the ECB. We calculate the average share of securities purchased under each of the programmes (the CBPP3, ABSPP and PSPP) over the first four months of the programme. We use the assumption that the distribution of the securities purchased under each programme will remain stable.

5. For the terminated programmes, we assume that ECB holdings decrease gradually to zero with the rate depending on the average maturity of the respective programme. We also assume that the ECB's PSPP holding will be depleted by September 2025 and LTROs will stabilise at a constant level. The future values of the process at the times $t+u, u=1,2,\ldots$ are assumed to depend on our projection for these holdings and GDP. Figure 1 shows the time series of each APP programme and LTROs from July 2009 to September 2015 and their projections after September 2015. In Figures 2 and 3 one can see the actual and projection values of the variables X_t^a and X_t^d.

4. ESTIMATION AND RESULTS

4.1 Assumption

As advocated by Krippner (2015), we choose model specification with two interest rate unobservable factors and impose the following normalising restrictions for the parameters identification: i) $\delta_1 = [1,1]'$; ii) $\delta_1 = 0$; iii) $\mu^{\mathbb{Q}} = 0$; iv) $\Phi^{\mathbb{Q}}$ is a diagonal matrix; v) $\sum$ is the lower triangular; and vi) $\lambda_0^1 = 0$. The model is estimated by the extended Kalman filter under the assumption that all yields are observed with measurement errors all having the same standard deviation σ. In the spirit of Christensen and Rudebusch (2013), we impose a near-unit-root restriction for the process X_t^1 under both probability measures to avoid small-sample estimation bias of less persistence in interest rate dynamics. We estimate the log-likelihood function over the grid of –20, –15, –10, –5 and 0 basis points and find its maximum at the $r^{LB} = -10$ basis points.

4.2 Results

The estimation results are reported in the Table.

Table

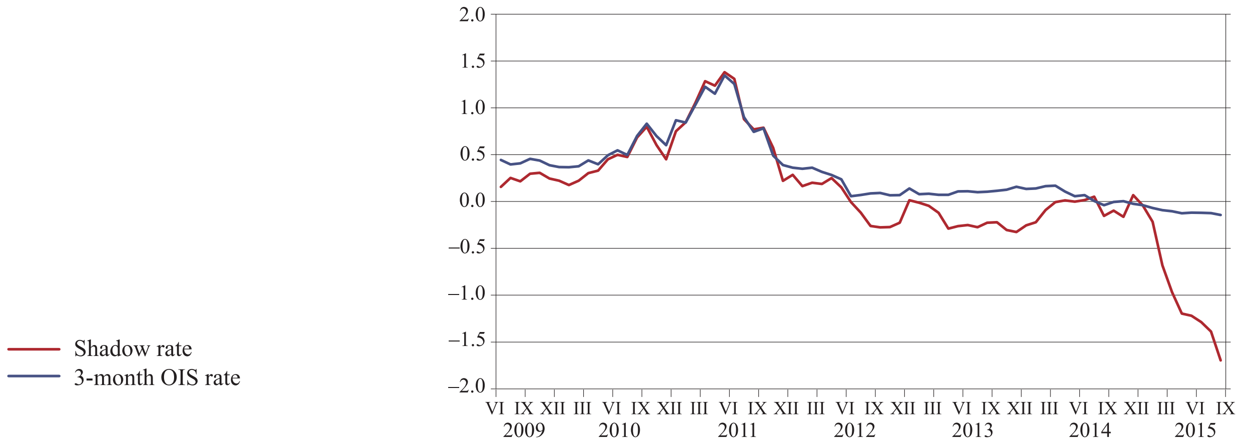
Maximum likelihood estimates for the period from July 2009 to September 2015

(asymptotic standard errors are given in parenthesis)

Coefficient	Value	
$\Phi^{\mathbb{Q}}$	$1 - 1 \cdot 10^{-7}$	0.9797
		(0.007)
λ_1	6.4754	96.2617
	(0.04)	(0.34)
	–72.8188	–20.2015
	(0.53)	(0.1)
λ_0	0.2831	
	(0.004)	
Σ	0.0009	–
	$(5 \cdot 10^{-6})$	
	–0.0010	0.0001
	$(1.6 \cdot 10^{-5})$	$(4.5 \cdot 10^{-7})$
σ	$6.73 \cdot 10^{-5}$	
	$(4 \cdot 10^{-7})$	
$\delta_{2,1}$	$-2.46 \cdot 10^{-5}$	
	$(4.46 \cdot 10^{-7})$	
$\delta_{2,2}$	$-2.69 \cdot 10^{-4}$	
	$(1 \cdot 10^{-6})$	

Figure 4 shows the time series of the computed shadow rate implied by formula (2) and the 3-month OIS rate. The form of the shadow rate is consistent with its other estimations based on euro area data (e.g., Lemke and Vladu (2014)). It moved into negative territory when on 26 July 2012 ECB President Mario Draghi made a pledge that "the ECB is ready to do whatever it takes to preserve the euro" within its mandate. The shadow rate started to decline significantly after January 2015 when the ECB announced its APP. At the end of the sample in September 2015, the shadow rate reached –1.69%.

Figure 4
Shadow rate and 3-month OIS rate
(%)



4.3 Effect of the ECB's balance sheet on the long-term interest rate

Figure 5 shows the actual and counterfactual dynamics of the euro area's 5-year OIS rate since June 2014 when the ECB reduced its deposit facility rate to a negative level. We choose a 5-year maturity as it captures the maturity preference of ECB purchases. The counterfactual values of the 5-year OIS rate are obtained from the model by assuming that the ECB balance sheet and its average maturity remain unchanged from July 2009. For the comparison purposes, the graph also contains the shadow rate, which can be treated as an indicator of the monetary policy stance.[3]

Figure 5
5-year OIS rate, counterfactual rate and shadow rate
(%)

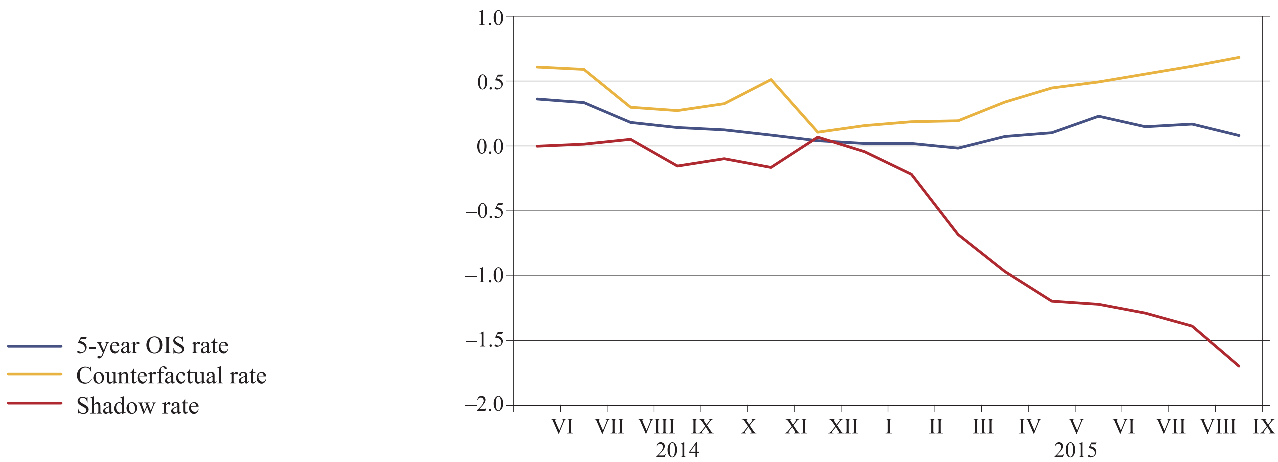


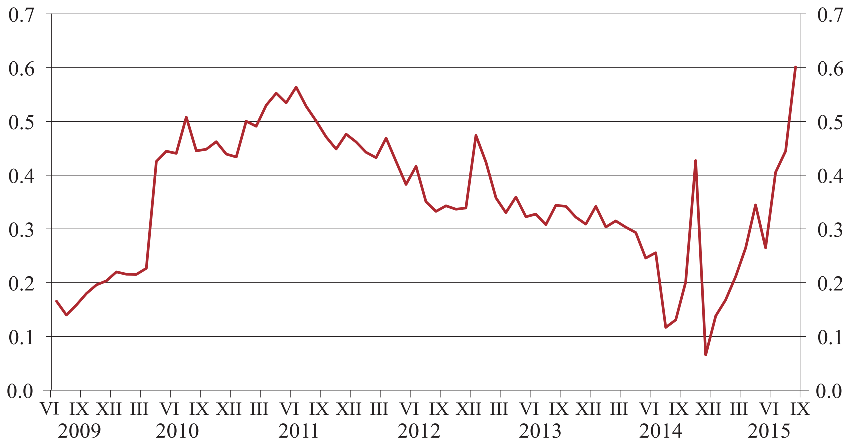
Figure 5 illustrates that the counterfactual rate started to diverge from the 5-year OIS rate as of January 2015. The model-based counterfactuals suggest that at the end of

[3] However, some studies (e.g. by Bauer and Rudebusch (2015) and Krippner (2015)) show that this is not a robust measure for the monetary policy stance.

September the 5-year OIS rate was about 60 basis points lower than it would otherwise be.

Figure 6 presents the spread between the counterfactual and actual 5-year OIS rates, which can be treated as a reduction in the interest rate due to the implementation of the non-standard policy measures. From early 2013 to September 2014, the size of the ECB's balance sheet declined progressively as a result of the possibility for banks to repay the 3-year LTROs. This led to a gradual decrease in the spread between the 5-year OIS rate and counterfactual rate to 12 basis points in August 2014. Figures 2 and 3 show that the ECB's balance sheet and its average maturity also declined during this period. The spread between the counterfactual and actual 5-year OIS rates began to increase rapidly from March 2015 when the PSPP was launched.

Figure 6
The spread between the counterfactual and actual 5-year OIS rates
(percentage points)



The large swing in November–December 2014 was driven by the reduction of the LTRO position in the ECB's balance sheet related to the substitution effect of the TLTRO that was just started. Furthermore, the ECB launched the ABSPP in November 2014. Despite the fact that the size of this programme is smaller than that of the PSPP, its duration at the start of the programme was above 30 years which is a much longer period than that of all other programmes. In our future work we plan to consider the characteristics of LTROs as separate factors, which can help to avoid such a large swing in the theoretical spread.

5. CONCLUSIONS

In response to the crisis, the ECB made an intensive use of the Eurosystem's balance sheet as a tool of the monetary policy in pursuit of its price stability mandate. The reasons behind this action were liquidity shortages and market impairments that impeded the transmission of the intended monetary policy stance as well as the need for further easing of the stance when the short-term interest rate reached its effective lower bound.

This paper proposes a term structure of an interest rate model that takes into account the ZLB and incorporates the euro amount of each APP programme and LTROs as well as their maturity structure.

We estimate the model by using the data sample from July 2009 to September 2015 and by employing the extended Kalman filter method. The model specification allows the systemic evaluation of the difference between the actual and counterfactual rates by explicitly modelling the effect of the APP and LTROs.

Our results show that the shadow rate moved into negative territory when ECB President Mario Draghi made a pledge on 26 July 2012 that the ECB, within its mandate, would do whatever it takes to preserve the euro and stabilise the rate at around –1.7% by September 2015. The 5-year OIS rate declined about 40 basis points at the start of the PSPP in March 2015, and at the end of September 2015 it was about 60 basis points lower than it would have been in the case of the absence of the APP.

The following topics may be considered as future developments of the proposed modelling approach: 1) estimation of the effective monetary stimulus by Krippner's alternative measure of the monetary policy stance instead of the shadow rate; 2) consideration of a more detailed division of the factor of the non-standard policy measures, e.g. by LTROs and the APP; and 3) the effect of anticipation of programmes before their official announcement (Altavilla et al. (2015)).

APPENDIX

A1 Conditional moments of the factors

Conditional expectations of the factors are:

$$E_t^{\mathbb{Q}} X_{t+i}^1 = (1-(\Phi^{\mathbb{P}})^i)\mu^{\mathbb{P}} + (\Phi^{\mathbb{P}})^i X_t^1 \tag{11}$$

Conditional expectations of the shadow rate are:

$$E_t^{\mathbb{Q}} s_{t+i} = \mu_{t,t+i} = \delta_0 + \delta_1 E_t^{\mathbb{Q}} X_{t+i}^1 + \delta_2 X_{t+i}^2 \tag{12}$$

The conditional variances are:

$$Var_t^{\mathbb{Q}}[s_{t+i}] = \sigma_{t,t+i}^2 = \delta_1' Var_t^{\mathbb{Q}}[X_{t+i}^1]\delta_1 \tag{13}$$

The conditional autocovariances are:

$$Cov_t^{\mathbb{Q}}[s_{t+i}, s_{t+j}] = \sigma_{t,t+i,t+j} = \delta_1' Cov_t^{\mathbb{Q}}[X_{t+i}^1, X_{t+j}^1]\delta_1 \tag{14}$$

A2 Conditional moments of the short rate

The expectations of the short rate are:

$$E_t[r_{t+i}] = E_t[max(0, s_{t+i})] = \mu_{t,t+i}\Phi\left(\frac{\mu_{t,t+i}}{\sigma_{t,t+i}}\right) + \sigma_{t,t+i}\phi\left(\frac{\sigma_{t,t+i}}{\mu_{t,t+i}}\right) \tag{15}$$

The autocovariances are:

$$\begin{aligned} E_t[r_{t+i}r_{t+j}] &= E_t[max(0,s_{t+i})max(0,s_{t+j})] \\ &= (\mu_{t,t+i}\mu_{t,t+j} + \sigma_{t,t+i,t+j})\Phi^d(-\zeta_{t,t+i}, -\zeta_{t,t+j}; \chi_{t,t+i,t+j}) \\ &+\sigma_{t,t+j}\mu_{t,t+i}\phi(\zeta_{t,t+j})\Phi\left(\frac{\zeta_{t,t+i}-\chi_{t,t+i,t+j}\zeta_{t,t+j}}{\sqrt{1-\chi_{t,t+i,t+j}^2}}\right) \\ &+\sigma_{t,t+i}\mu_{t,t+j}\phi(\zeta_{t,t+i})\Phi\left(\frac{\zeta_{t,t+j}-\chi_{t,t+i,t+j}\zeta_{t,t+i}}{\sqrt{1-\chi_{t,t+i,t+j}^2}}\right) \\ &+\sigma_{t,t+i}\sigma_{t,t+j}\sqrt{\frac{1-\chi_{t,t+i,t+j}^2}{2\pi}}\,\phi\left(\sqrt{\frac{\zeta_{t,t+i}^2-2\chi_{t,t+i,t+j}\zeta_{t,t+i}\zeta_{t,t+j}\zeta_{t,t+j}^2}{1-\chi_{t,t+i,t+j}^2}}\right) \end{aligned} \tag{16}$$

where $\zeta_{t,t+i} = \frac{\mu_{t,t+i}}{\sigma_{t,t+i}}$, $\chi_{t,t+i,t+j} = \frac{\sigma_{t,t+i,t+j}}{\sigma_{t,t+i}\sigma_{t,t+j}}$ and ϕ are the univariate standard normal pdf, Φ is the univariate standard normal cdf and Φ_2 is the cumulative bivariate Gaussian distribution function with correlation $\chi_{t,t+i,t+j}$, Φ^d denotes decumulative bivariate Gaussian distribution function, in particular:

$$\Phi^d(\zeta_{t+i}, \zeta_{t+j}; \chi_{t,t+i,t+j}) = 1 - \Phi(\zeta_{t+i}) - \Phi(\zeta_{t+j}) + \Phi_2(\zeta_{t+i}, \zeta_{t+j}; \chi_{t,t+i,t+j})).$$

For $i = j$ formula (16) can be simplified

$$E_t[r_{t+i}^2] = (\mu_{t,t+i}^2 + \sigma_{t,t+i}^2)\Phi(\zeta_{t,t+i}) + \mu_{t,t+i}\sigma_{t,t+i}\phi\zeta_{t,t+i} \tag{17}$$

A3 Bond prices and yields

The price of an n-period zero-coupon bond P_t^n is given by

$$P_t^n = E_t^{\mathbb{Q}} \exp(-\sum_{i=0}^{n-1} r_{t+i}) = \exp(-r_t) E_t^{\mathbb{Q}} \exp(-\sum_{i=1}^{n-1} r_{t+i}) \quad (18).$$

Assuming that $\sum_{i=0}^{n-1} r_{t+i}$ is small and expanding exponentially in powers of this sum up to order two, we have

$$P_t^n = \exp(-r_t) E_t^{\mathbb{Q}} \left[1 - \sum_{i=1}^{n-1} r_{t+i} + \frac{1}{2}(\sum_{i=1}^{n-1} r_{t+i})^2\right] \quad (19)$$

or

$$P_t^n = \exp(-r_t)\left(1 - \sum_{i=1}^{n-1} E_t^{\mathbb{Q}} r_{t+i} + \frac{1}{2}\left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} E_t^{\mathbb{Q}} r_{t+i} r_{t+j}\right)\right) \quad (20).$$

Specifically, for $n = 2$ we have

$$P_t^2 = \exp(-r_t)\left(1 - E_t^{\mathbb{Q}} r_{t+1} + \frac{1}{2} E_t^{\mathbb{Q}} r_{t+1}^2\right) \quad (21)$$

and for $n = 3$

$$P_t^3 = \exp(-r_t)\left[1 - E_t^{\mathbb{Q}}(r_{t+1} + r_{t+2}) + \frac{1}{2} E_t^{\mathbb{Q}}(r_{t+1}^2 + 2 r_{t+1} r_{t+2} + r_{t+2}^2)\right] \quad (22).$$

Formula (20) can be written in a recursive way

$$P_t^n = P_t^{n-1} \exp(-r_t)\left(-E_t^{\mathbb{Q}} r_{t+n-1} + \sum_{i=1}^{n-1} E_t^{\mathbb{Q}} r_{t+i} r_{t+n-1} + \frac{1}{2} E_t^{\mathbb{Q}} r_{t+n-1}^2\right) \quad (23).$$

From (20) we have the approximate measurement equation for yields

$$y_t^n = -\frac{1}{n} \ln(P_t^n) = g_n^{app}(X_t, \theta) \quad (24),$$

where

$$g_n^{app}(X_t, \theta) = -\frac{1}{n}\left\{-r_t + \ln\left[1 - \sum_{i=1}^{n-1} E_t^{\mathbb{Q}} r_{t+i} + \frac{1}{2}\left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} E_t^{\mathbb{Q}} r_{t+i} r_{t+j}\right)\right]\right\} \quad (25).$$

A4 The extended Kalman filter setup

The transition is in (1). From (24) the measurement equations have the form:

$$y_t^n = g_n^{app}(X_t^1, \theta) + \eta_{t+1}^n \quad (26),$$

where η_{t+1}^n is a measurement error corresponding to the yield of a maturity n.

Measurement equation (24) is non-linear with respect to the state variables X_t; hence, we need to use the extended Kalman filter. To obtain it, we linearise equation (24) at the point $X_{t|t-1}^1$, where $X_{t|t-1}^1 = (1 - \Phi^{\mathbb{P}})\mu^{\mathbb{P}} + \Phi^{\mathbb{P}} X_{t-1|t-1}^1$ is a one-step-ahead state prediction at the time moment t_{i-1}

$$g_n^{app}(X_t^1, \theta) \approx g_n^{app}\left(X_{t|t-1}^1, \theta\right) + \frac{\partial g_n^{app}(X_t^1, \theta)}{\partial X}\left(X_t^1 - X_{t|t-1}^1\right).$$

The derivative is:

$$\frac{\partial g_n^{app}(X_t^1, \theta)}{\partial X} = -\frac{1}{n} \frac{\partial \ln(P_t^n)}{\partial X} = -\frac{1}{n} \frac{\partial (P_t^n)}{\partial X} \frac{1}{P_t^n}$$

$$E_t^{\mathbb{Q}} X_{t+i}^1 = (1 - (\Phi^{\mathbb{P}})^i)\mu^{\mathbb{P}} + (\Phi^{\mathbb{P}})^i X_t^1 \quad (27).$$

A5 Construction of the variable X_t^b

$$X_t^b = \begin{cases} \dfrac{(LTRO_t + CBPP_t)}{GDP_t}, 2009M7 - 2010M4; \\ \dfrac{(LTRO_t + CBPP_t + SMP_t)}{GDP_t}, 2010M5 - 2010M10; \\ \dfrac{(LTRO_t + CBPP_t + SMP_t + CBPP2_t)}{GDP_t}, 2010M11 - 2014M8; \\ \dfrac{(LTRO_t + CBPP_t + SMP_t + CBPP2_t + TLTRO_t)}{GDP_t}, 2014M9; \\ \dfrac{(LTRO_t + CBPP_t + SMP_t + CBPP2_t + TLTRO_t + CBPP3_t)}{GDP_t}, 2014M10; \\ (LTRO_t + CBPP_t + SMP_t + CBPP2_t + TLTRO_t + CBPP3_t \\ +ABSPP_t)\dfrac{1}{GDP_t}, 2014M11 - 2015M2; \\ (LTRO_t + CBPP_t + SMP_t + CBPP2_t + TLTRO_t + CBPP3_t + ABSPP_t \\ +PSPP_t)/GDP_t, 2015M3 - 2015M9. \end{cases}$$

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